Signal Basics

Hongkai Xiong

熊红凯

http://min.sjtu.edu.cn

Department of Electronic Engineering

Shanghai Jiao Tong University

Signals

- **Definition** **[The American Heritage Dictionary of the English Language]**

  a. *Electronics* An impulse or fluctuating quantity, as of electrical voltage or light intensity, whose variations represent coded information.
  b. *Computers* A sequence of digital values whose variations represent coded information.

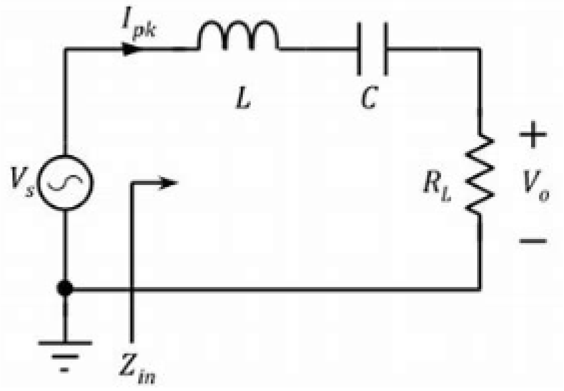
- **Examples**
  - voltages or currents in circuits
  - speech, images, videos

- **Mathematical Representation**
  - Function of one or more independent variables

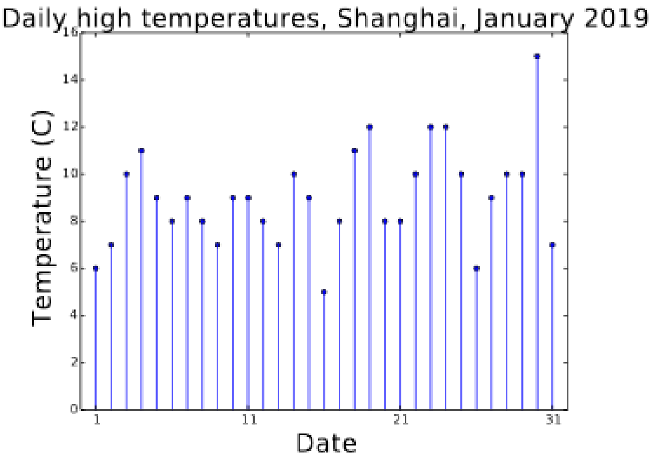
$$x: I \to X$$
$$t \mapsto x(t)$$

Examples of Signals

- **Electrical voltage**
  - $V_0 : \mathbb{R} \to \mathbb{R}$
    $t \mapsto V_0(t)$



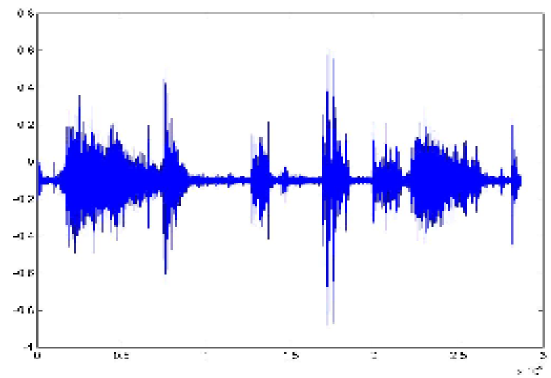
- **Daily temperature**
  - $T : I \to \mathbb{R}$
    $n \mapsto T[n]$



Examples of Signals

- **Speech signal**
  - $x: \mathbb{R} \to \mathbb{R}$

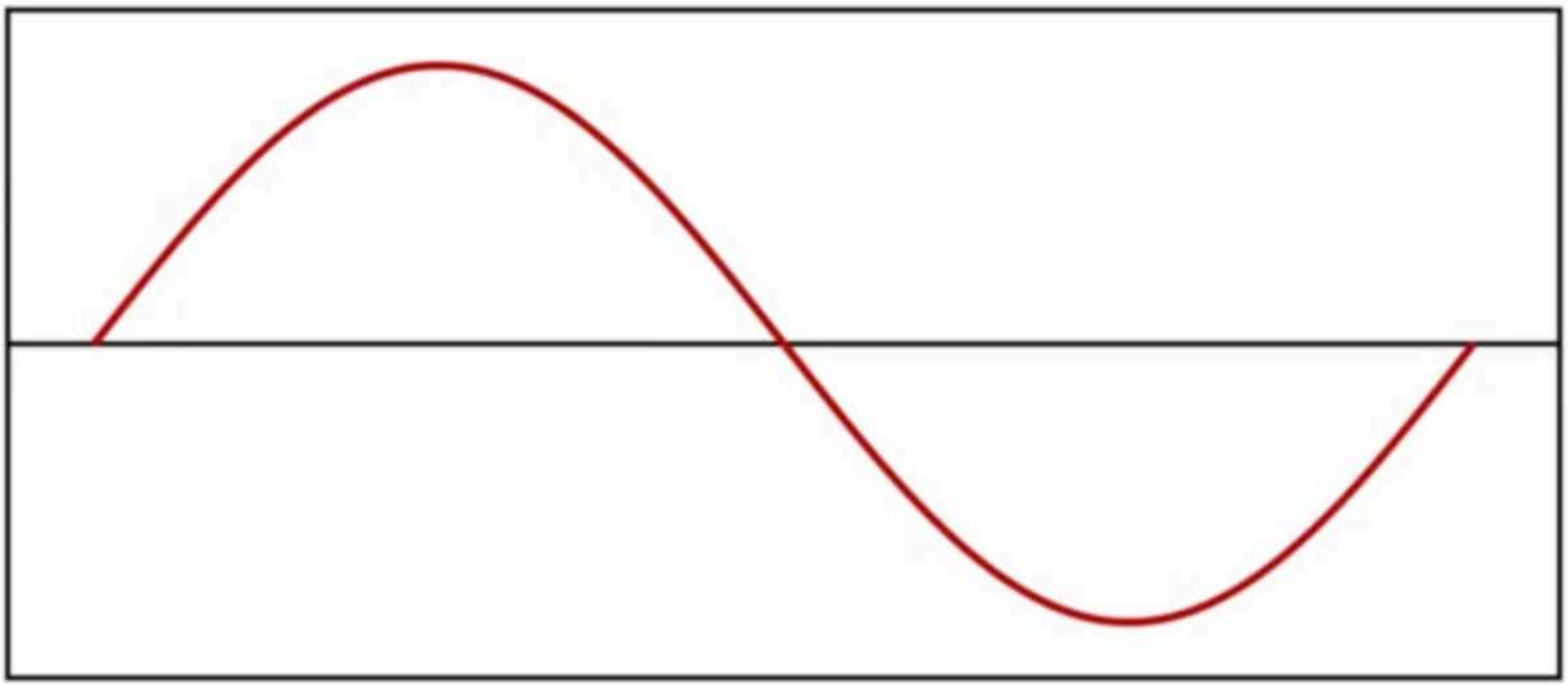
    $t \mapsto x(t)$



- **Color image**
  - $P: I \times J \to R \times G \times B$

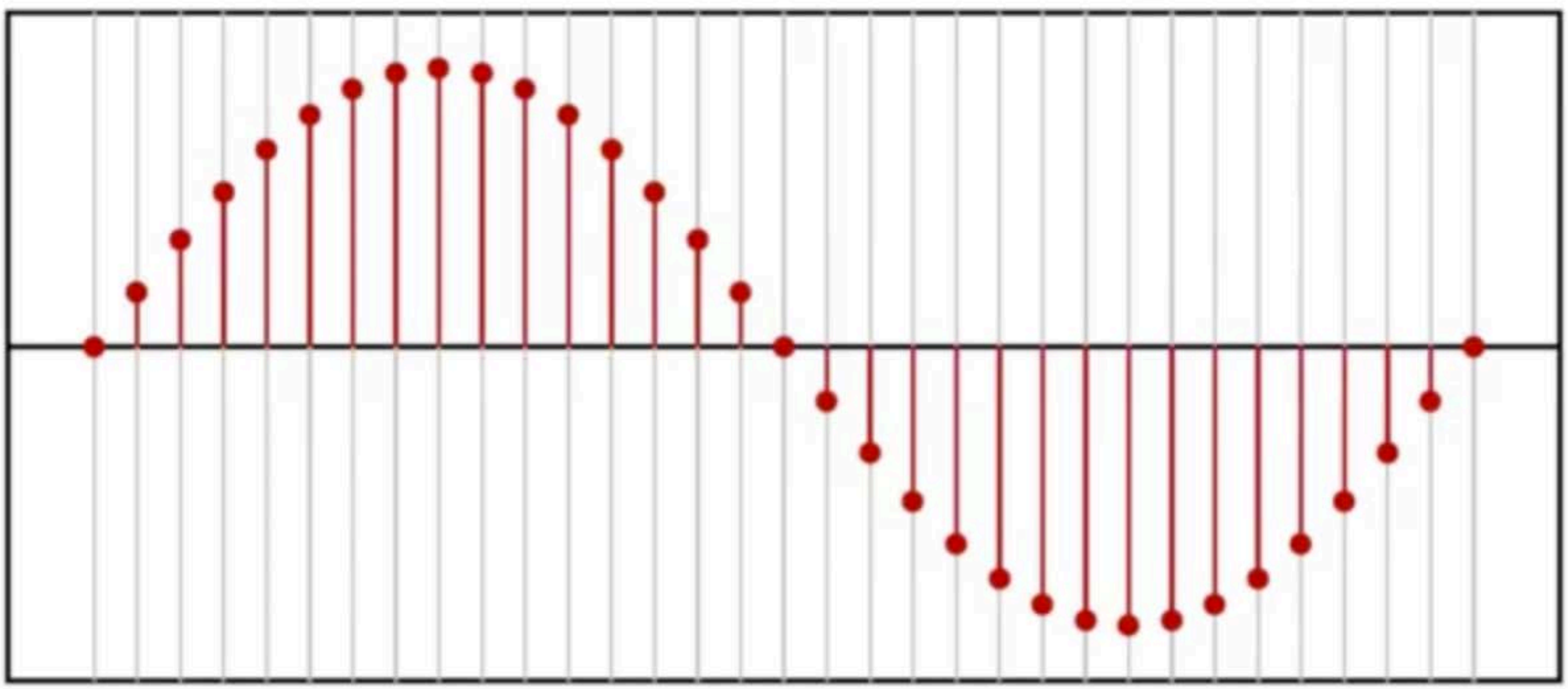
    $(i,j) \mapsto (r[i,j], g[i,j], b[i,j])$

- **Continuous-time signal (Analog signal)**



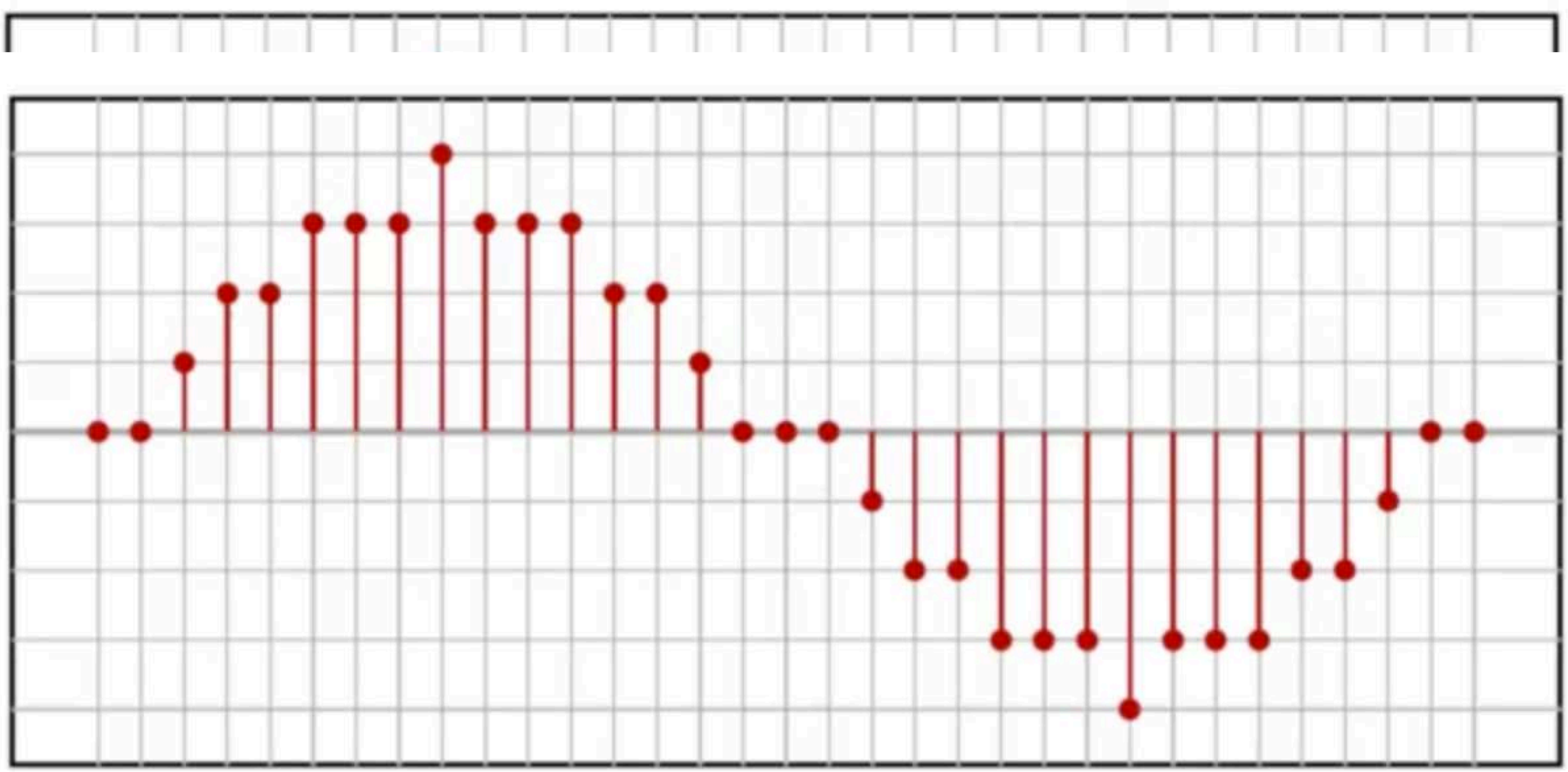
$x(t)$

- **Discrete-time signals sampled from continuous-time signals**



$x[n]$

- **Digital signals from continuous-time signals (Analog signal)**



$\hat{x}[n]$

- **Digital signals from continuous-time signals (Analog signal)**

- **The world is analog, the computer is digital**

- **Independent variables can be**
  - continuous, e.g.,
    - voltage/current
    - vehicle speed
  - discrete, e.g.,
    - DNA base sequence
    - weekly average for stock markets
  - 1-D, 2-D, ••• *n*-D, e.g.,
    - 2-D/3-D Digital image pixels

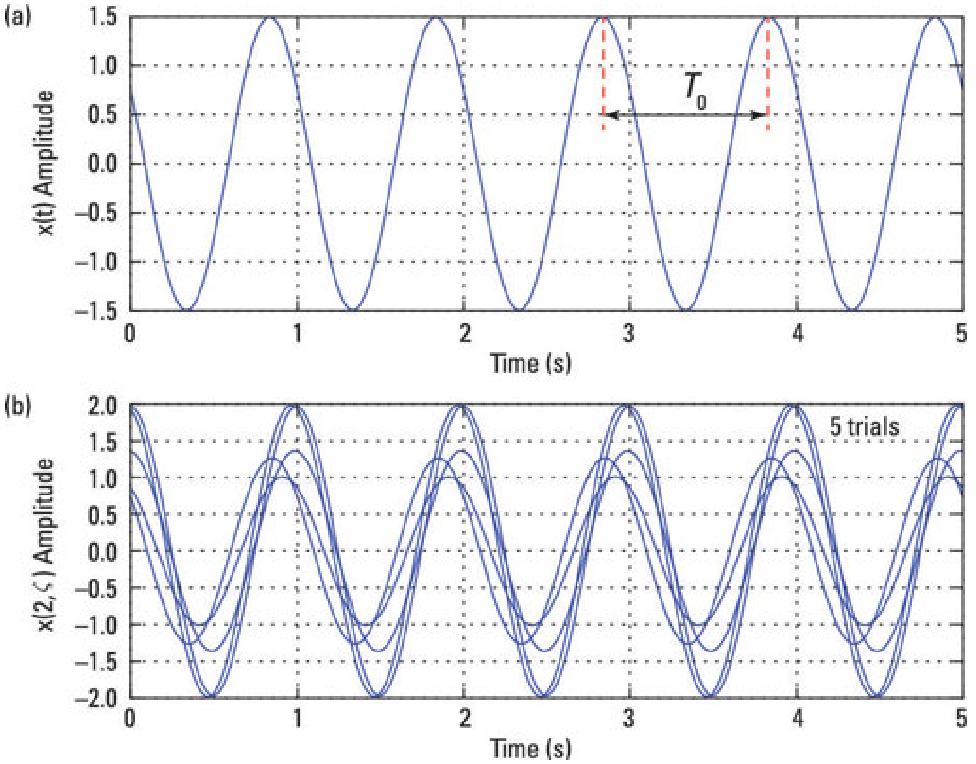
Classification of Signals

- **Deterministic signals vs. Random signals**
- **Continuous signals vs. Discrete signals**
- **Energy signals vs. Power signals**
- **Periodic signals vs. Non-periodic signals**
- **Odd signals vs. Even signals**
- **Real signals vs. Complex signals**
- **……**

Deterministic vs. Random Signals

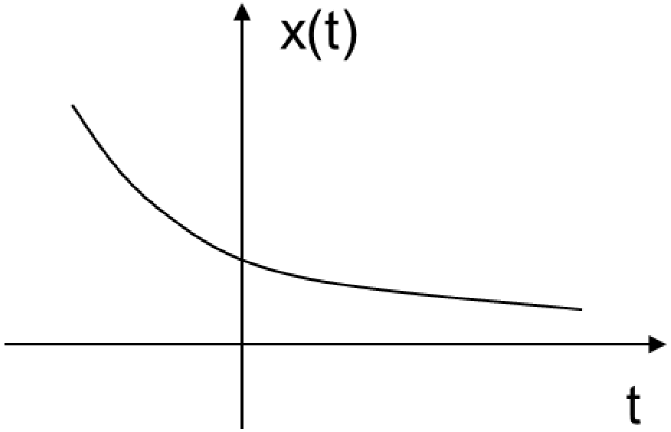
- **Deterministic signals**
  - can be described by exact mathematical expression
  - given t and get deterministic result

- **Random signals**
  - can not be described by exact mathematic expression
  - given t and get random result



Continuous-time vs. Discrete-time Signals

- **Continuous-time (CT) signals**
  - Independent variable t is continuous
  - The signal is defined for a continuum of values of the independent variable t
  - Notation: parentheses for continuous time, e.g., (t)

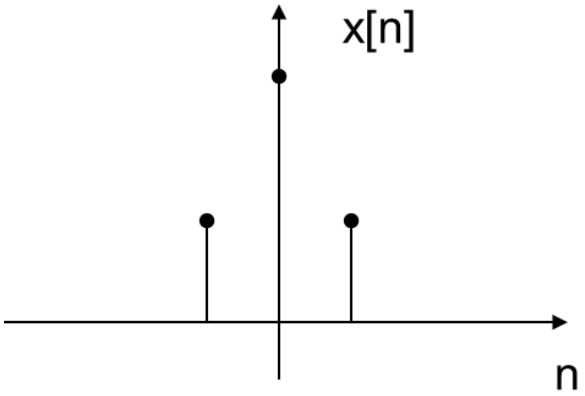


example: $x(t) = 2e^{-t}$

Continuous-time vs. Discrete-time Signals

- **Discrete-time (DT) signals/Sequences: $x[n]$**
  - Independent variable n takes on only a discrete set of values, (in this course, a set of integer values only)
  - Signal is defined only at discrete times
  - Notation: square brackets for discrete time, e.g., $[n]$

example: $x[n] = \begin{cases} 2, n = -1 \\ 4, n = \ 0 \\ 2, n = \ 1 \\ 0, others \end{cases}$



Power and Energy Signals

- **Power and energy in a physical system**
  - Instantaneous power

$$P(t) = v(t)\boldsymbol{i}(t) = \frac{1}{R}|v(t)|^2$$

  - Total energy over time interval $[t_1, t_2]$

$$\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} p(t)dt = \frac{1}{t_2 - t_1}\frac{1}{R}\int_{t_1}^{t_2} |v(t)|^2 dt$$

  - Average power over time interval $[t_1, t_2]$

$$\int_{t_1}^{t_2} p(t)dt = \frac{1}{R}\int_{t_1}^{t_2}|v(t)|^2 dt$$

上海交通大學
SHANGHAI JIAO TONG UNIVERSITY

Power and Energy Signals

- **Power and energy in this course**
  - Total Energy

$$E = \int_{t_1}^{t_2} |x(t)|^2 \mathrm{d}t, \qquad E = \sum_{n=n_1}^{n_2} |x[n]|^2$$

  - Average Power

$$P = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |x(t)|^2 dt, \quad P = \frac{1}{n_2 - n_1 + 1} \sum_{n=n_1}^{n_2} |x[n]|^2$$

Power and Energy Signals

- **Power and energy definitions over infinite interval**
  - Total Energy

$$E_\infty = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt \qquad E_\infty = \lim_{N\to\infty} \sum_{n=-N}^{N} |x[n]|^2$$

  - Average Power

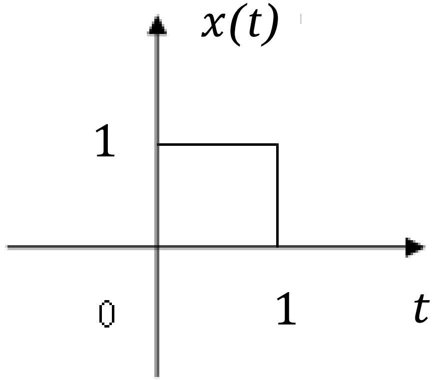
$$P_\infty = \frac{1}{2T} \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt \qquad P_\infty = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

Finite-energy and Finite-power Signals

- Finite-Energy Signal

$E_\infty < \infty \qquad P_\infty = 0$

example



- Finite-Average Power Signal

$P_\infty < \infty \qquad E_\infty = \infty$

example: $x[n]=4$

Periodic vs. Non-periodic Signals

- **Definition for continuous-time signals**
  - If $x(t) = x(t + T)$ for all values of t, then $x(t)$ is periodic, and $x(t) = x(t + mT)$ for all t and any integer m.
  - Fundamental period $\triangleq$ the smallest positive value that satisfies $x(t) = x(t + T)$ for all t

Question: if the signal is constant, what is the fundamental period ?

Periodic vs. Non-periodic Signals

- **Definition for discrete-time signals**
  - If $x[n]=x[n+N]$ for all values of n, then $x[n]$ is periodic, and $x[n]=x[n+mN]$ for all n and any integer m
  - Fundamental Period ≜ the smallest positive integer that satisfies $x[n]=x[n+N]$ for all n

Question: if the signal is constant, what is the fundamental period ?

Even vs. Odd Signals

- **Definition**
  - $x(t)$ or $x[n]$ is even if it is identical to its time-reversed counterpart

$$x(t) = x(-t) \qquad\qquad x[n] = x[-n]$$

  - Similarly, $x(t)$ or $x[n]$ is odd if

$$x(t) = -x(-t) \qquad\qquad x[n] = -x[-n]$$

Question: for odd signal $x(t)$, can we determine $x(0)$?

Even vs. Odd Signals

- **Even-odd decomposition of a signal**

$$x(t) = E_v\{x(t)\} + O_d\{x(t)\}$$

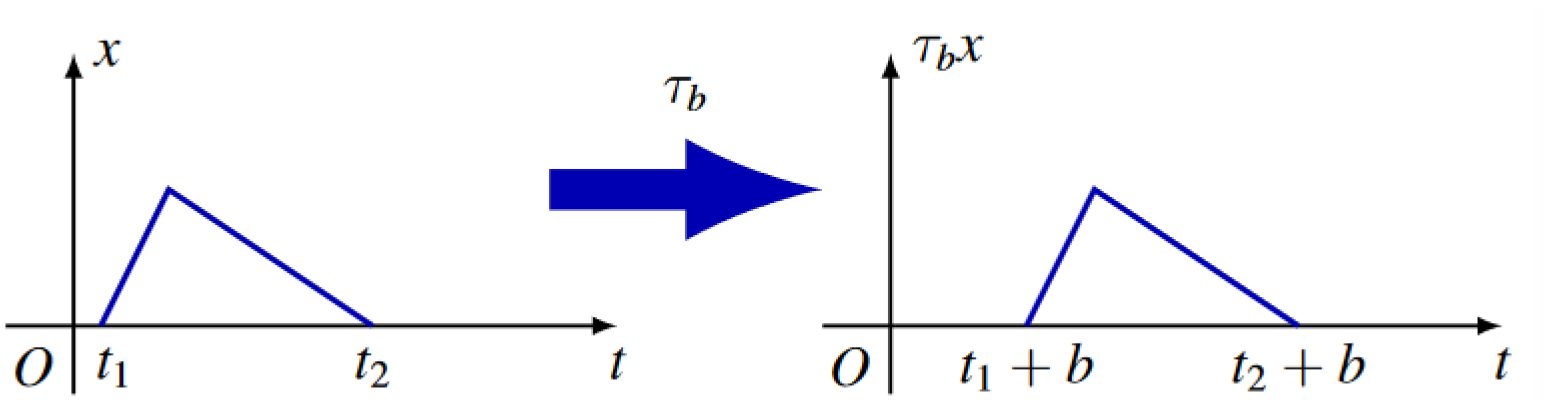
Even part **Odd part**

$$E_V\{x(t)\} = \frac{1}{2}[x(t) + x(-t)]$$

$$O_d\{x(t)\} = \frac{1}{2}[x(t) - x(-t)]$$

Time Shift

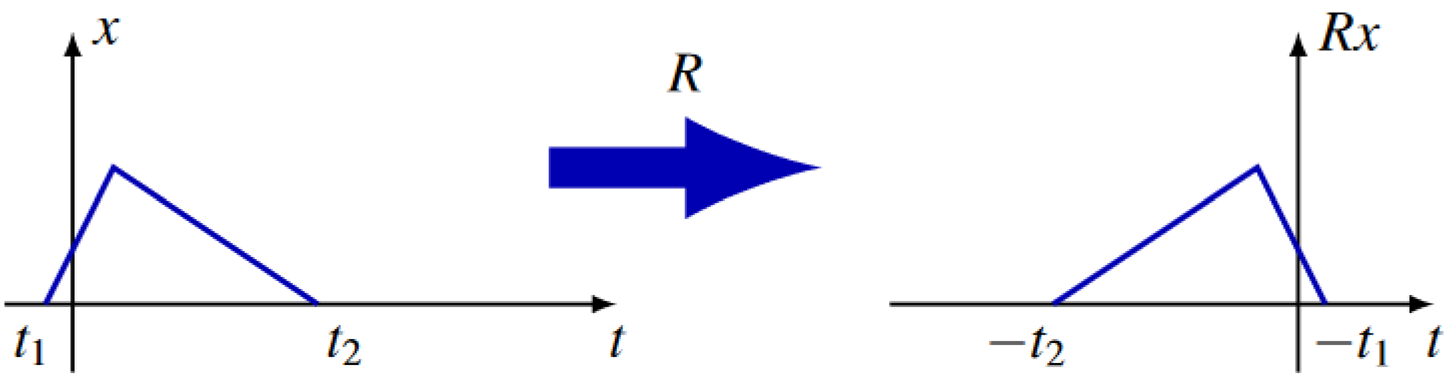
$$x(t) \to x(t-b) \qquad x[n] \to x[n-b]$$



- **Examples: radar, sonar, radio propagations**
- **Notes: each point in $x(t)/x[n]$ occurs at a later/early time in $x(t-t_0)/x[n-n_0]$, when t_0/n_0 is positive/negative, i.e.,**
  - $x(t-t_0)/x[n-n_0]$ is the delayed version of $x(t)/x[n]$, for $t_0/n_0 > 0$
  - $x(t-t_0)/x[n-n_0]$ is the advanced version of $x(t)/x[n]$, $for\ t_0/n_0 < 0$

Time Reversal

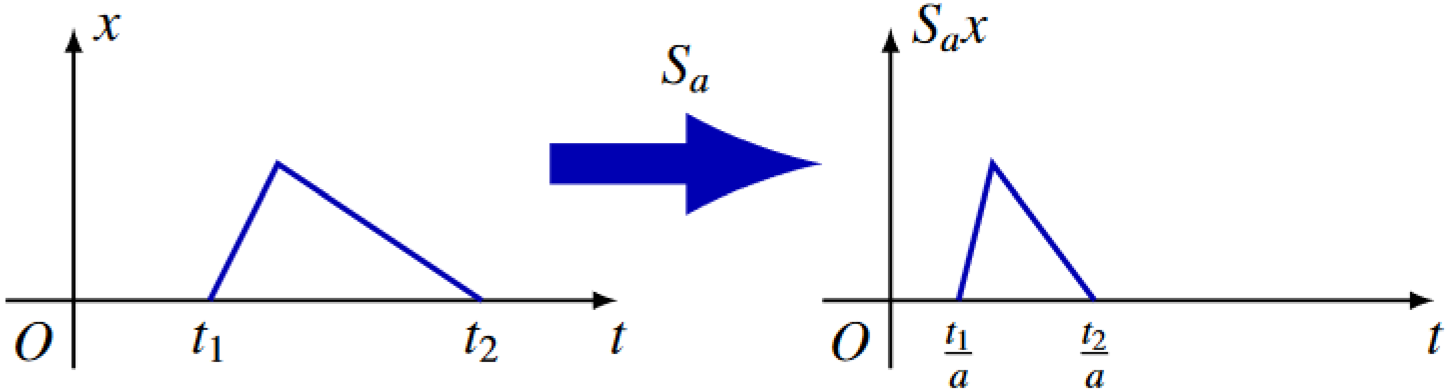
- **CT signal:** $x(t) \rightarrow x(-t)$



- **DT signal:** $x[n] \rightarrow x[-n]$
- **Example: tape recording played backward**

Time Scaling

- **Time Scaling:** $x(t) \to x(\alpha t)$, $x[n] \to x[\alpha n]$



- **Example: audio played back at different speed**
  - fast forward $\alpha > 1$
  - slow forward $0 < \alpha < 1$
  - slow backward $-1 < \alpha < 0$
  - fast backward $\alpha < -1$

Notes: $|\alpha| > 1$, Compression
$|\alpha| < 1$, Extension

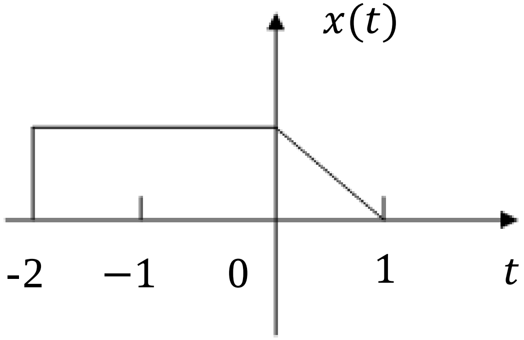
General Affine Transformation

- **Affine Transformation**

  $$x(t) \to x(\alpha t + \beta), \qquad x[n] \to x[\alpha n + \beta]$$

- **Examples:**

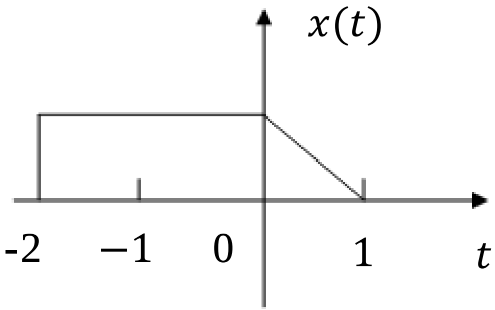
  $$x(t) \to x(-3t - 2)$$

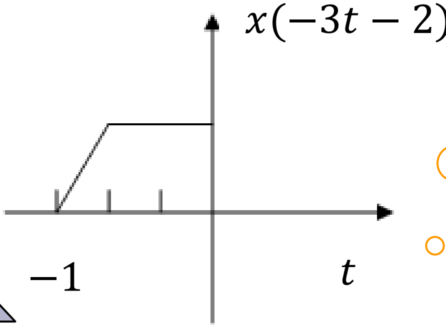


- **Can be decomposed as product of time shift, reversal and scaling, with the general rule:**
  - Time shift first
  - Then time reversal and time scaling

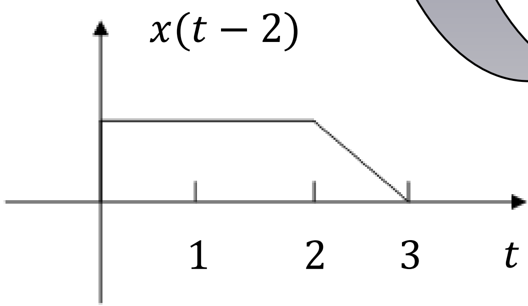
- **Examples:** $x(t) \rightarrow x(-3t-2)$

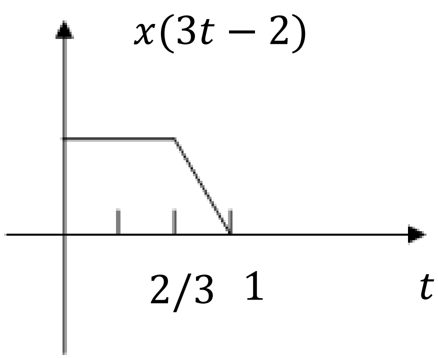
Question: what would happen if we shift after scaling/reflection?





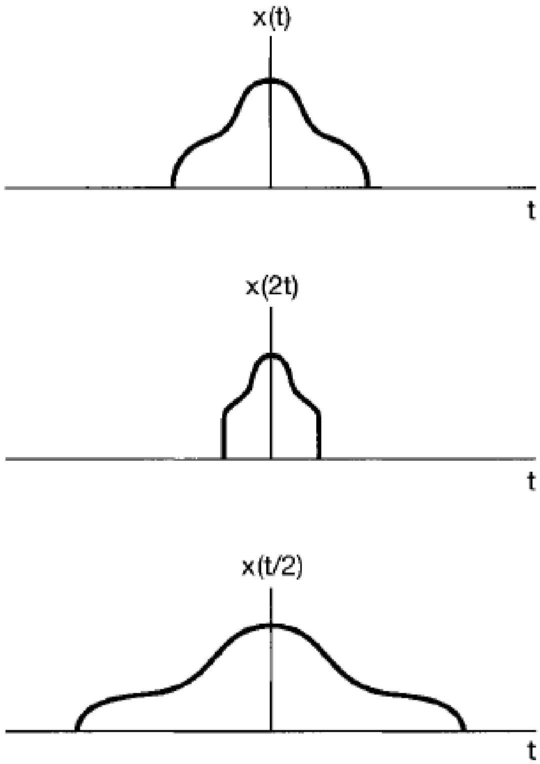
$x(-3(t+2/3))$

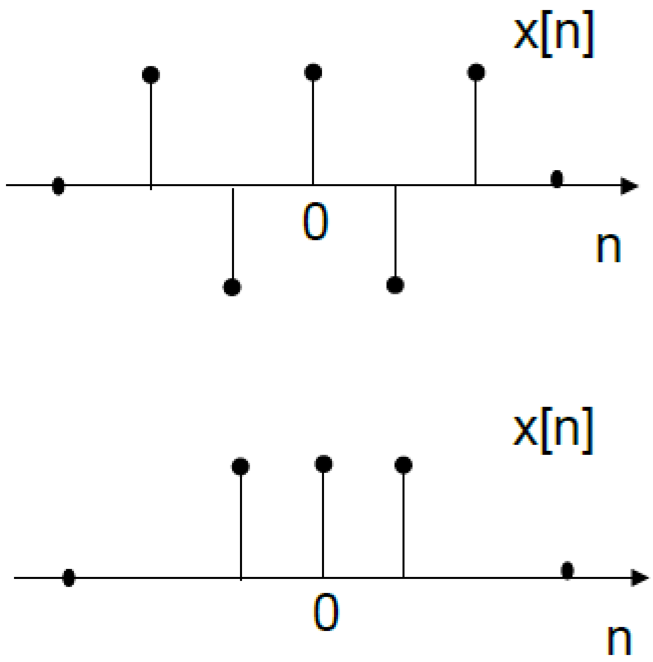




- **Examples:**

$$x(t) \to x(2t), x(\frac{t}{2}) \qquad x[n] \to x[2n]$$



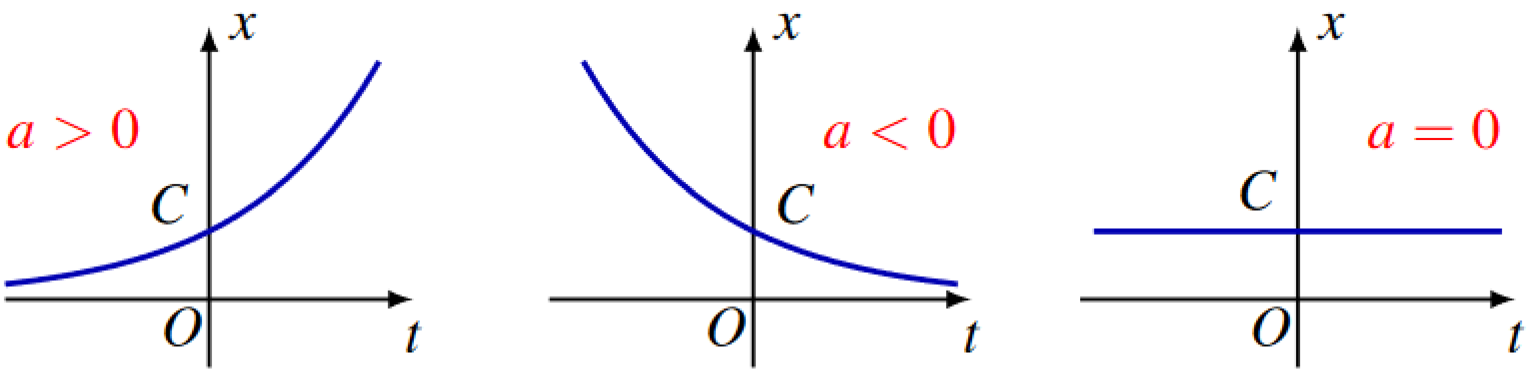


上海交通大學
SHANGHAI JIAO TONG UNIVERSITY

CT Complex Exponential Signals

$$x(t) = C \cdot e^{at}, \text{ where } C \in \mathbb{C}, a \in \mathbb{C}$$

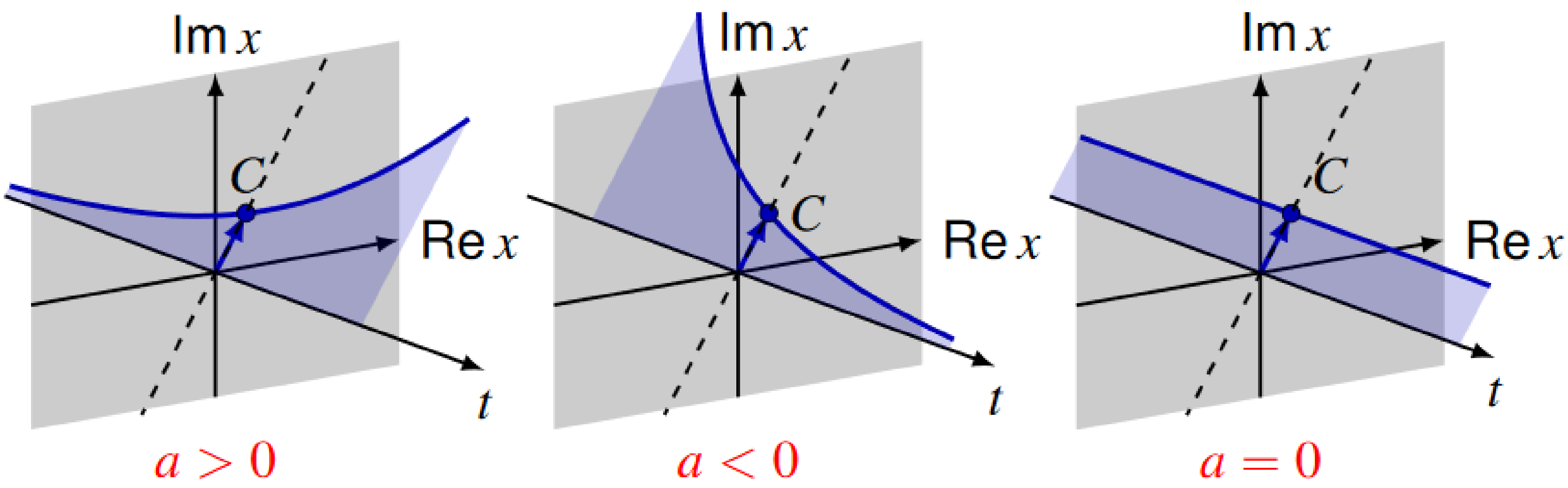
- **Real exponential signals:** $C \in \mathbb{R}, \boldsymbol{a} \in \mathbb{R}$
  - $a > 0$: growing exponential
  - $a < 0$: decaying exponential
  - $a = 0$: constant



CT Complex Exponential Signals

$$x(t) = C \cdot e^{at}, \text{ where } C \in \mathbb{C}, a \in \mathbb{R}$$

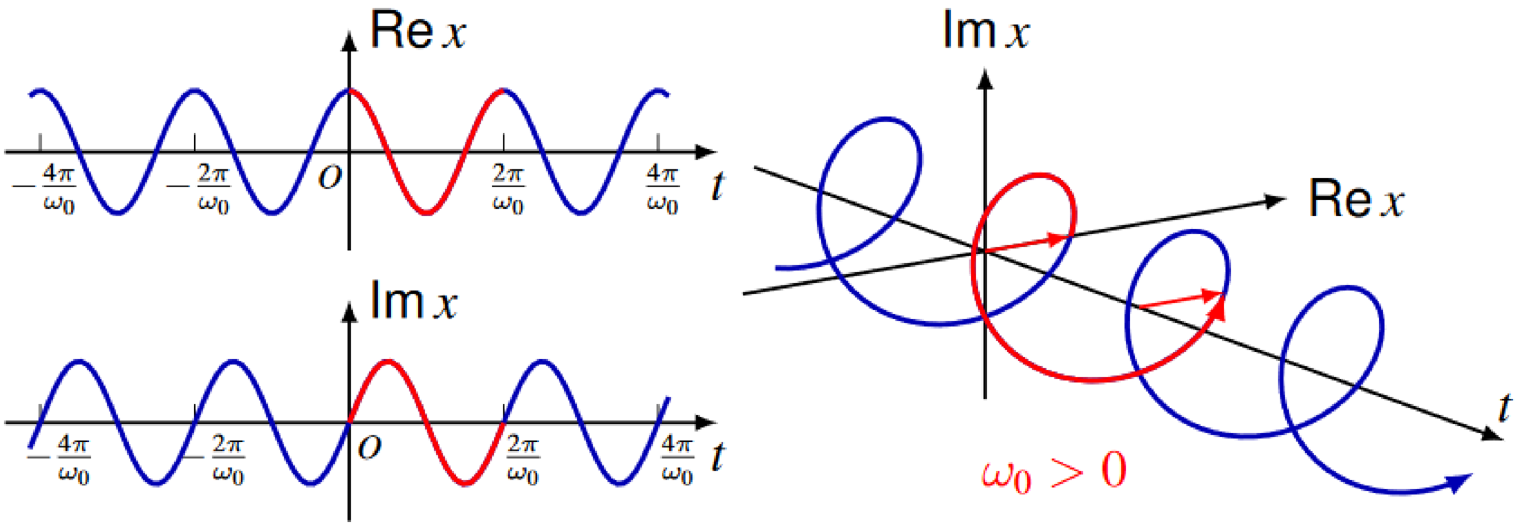
- $a > 0$: diverges from t axis, $|x(t)| \nearrow \infty$ as $t \to \infty$
- $a < 0$: converges to t axis, $|x(t)| \searrow \infty$ as $t \to \infty$
- $a = 0$: constant



Periodic Complex Exponential Signals

- **Euler's Relation:**

$$x(t) = e^{j\omega_0 t} = \cos(\omega_0 t) + j\sin(\omega_0 t), \text{ where } \omega_0 \in \mathbb{C}$$

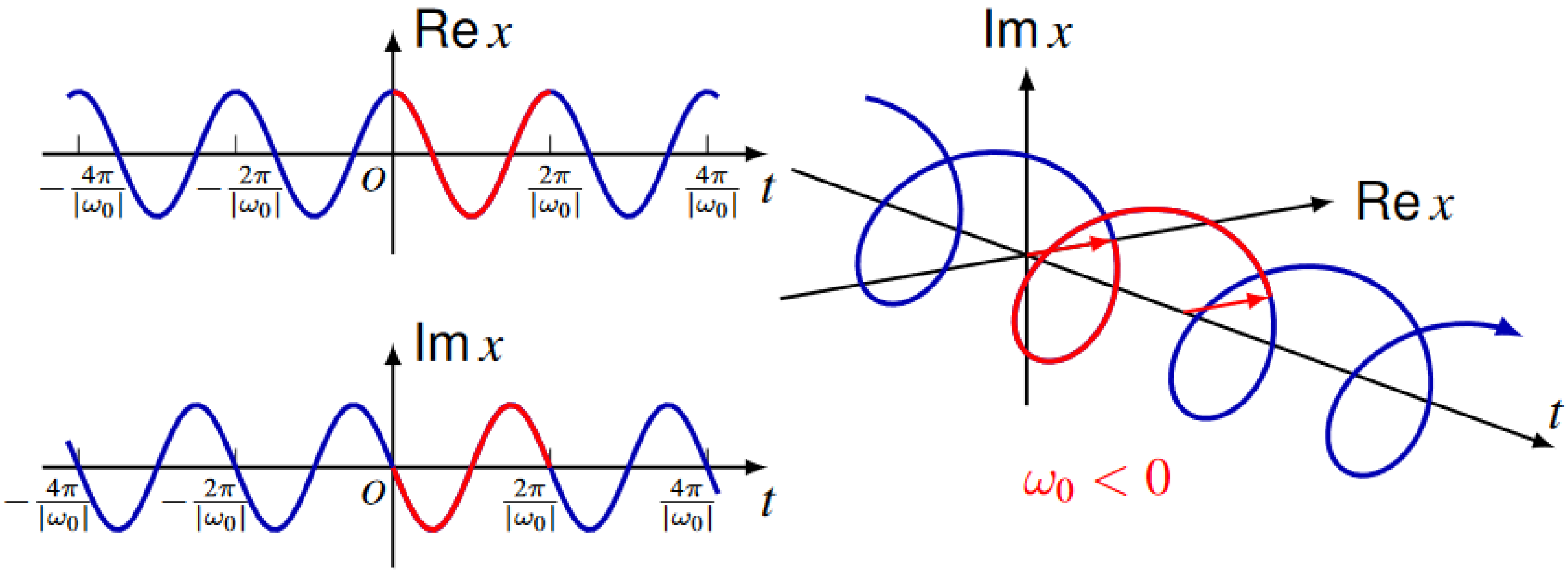


- (Angular) frequency: ω_0 radians/s
- Frequency: $f_0 = \omega_0/(2\pi)$ cycles/s (Hz)
- Fundamental period: $T_0 = 2\pi/|\omega_0| = 1/|f_0|$ s (only if $\omega_0 \neq 0$)

上海交通大學
SHANGHAI JIAO TONG UNIVERSITY

Periodic Complex Exponential Signals

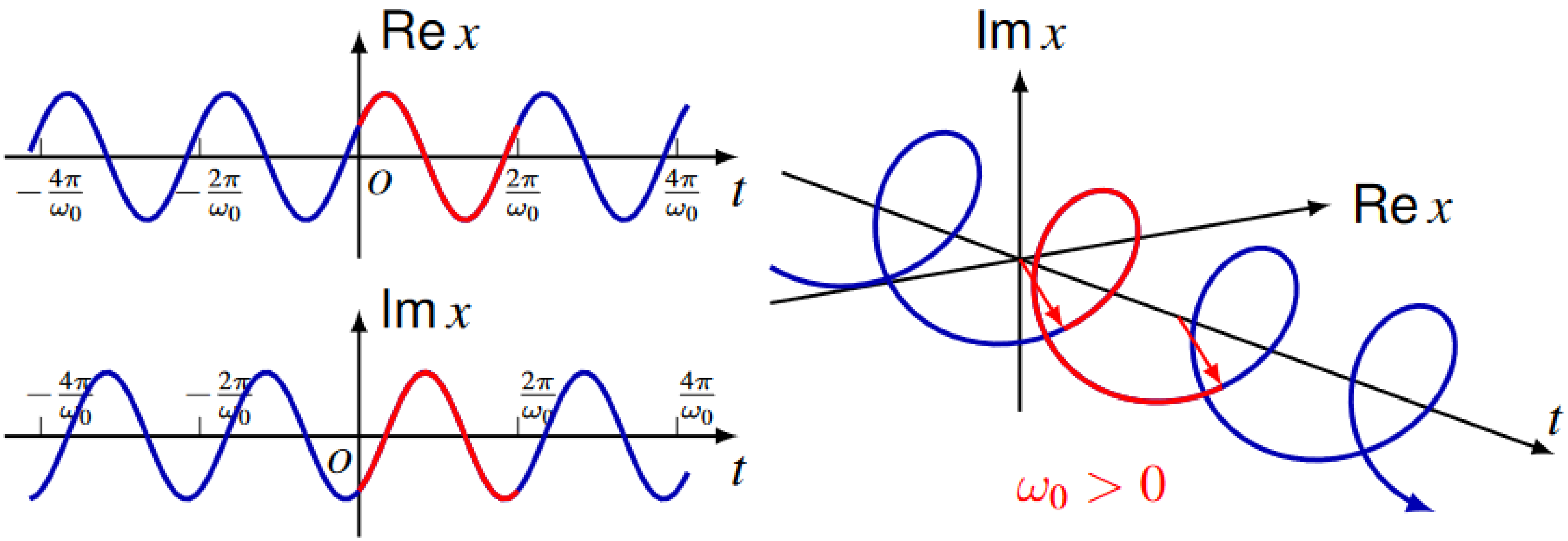
$$x(t) = e^{j\omega_0 t} = \cos(|\omega_0|\, t) - j\sin(|\omega_0|\, t), \text{ where } \omega_0 < 0$$



- Fundamental frequency: $|\omega_0|$, $|f_0|$

Periodic Complex Exponential Signals

$$x(t) = Ce^{j\omega_0 t} = |C|[\cos(\omega_0 t + \phi) + j\sin(\omega_0 t + \phi)],\ C = Ce^{j\phi}$$



CT Complex Sinusoidal Signals

$$x(t) = A\cos(\omega_0 t + \phi), \text{ where } A \in \mathbb{R}$$

- Conversion between exponentials and sinusoids

$$Ae^{j(\omega_0 t+\phi)} = A\cos(\omega_0 t + \phi) + jA\sin(\omega_0 t + \phi)$$

$$A\cos(\omega_0 t + \phi) = \frac{A}{2}e^{j\phi}e^{j\omega_0 t} + \frac{A}{2}e^{-j\phi}e^{-j\omega_0 t} = A \cdot \mathcal{Re}\{e^{j(\omega_0 t+\phi)}\}$$

$$A\sin(\omega_0 t + \phi) = A \cdot \mathcal{Im}\{e^{j(\omega_0 t+\phi)}\}$$

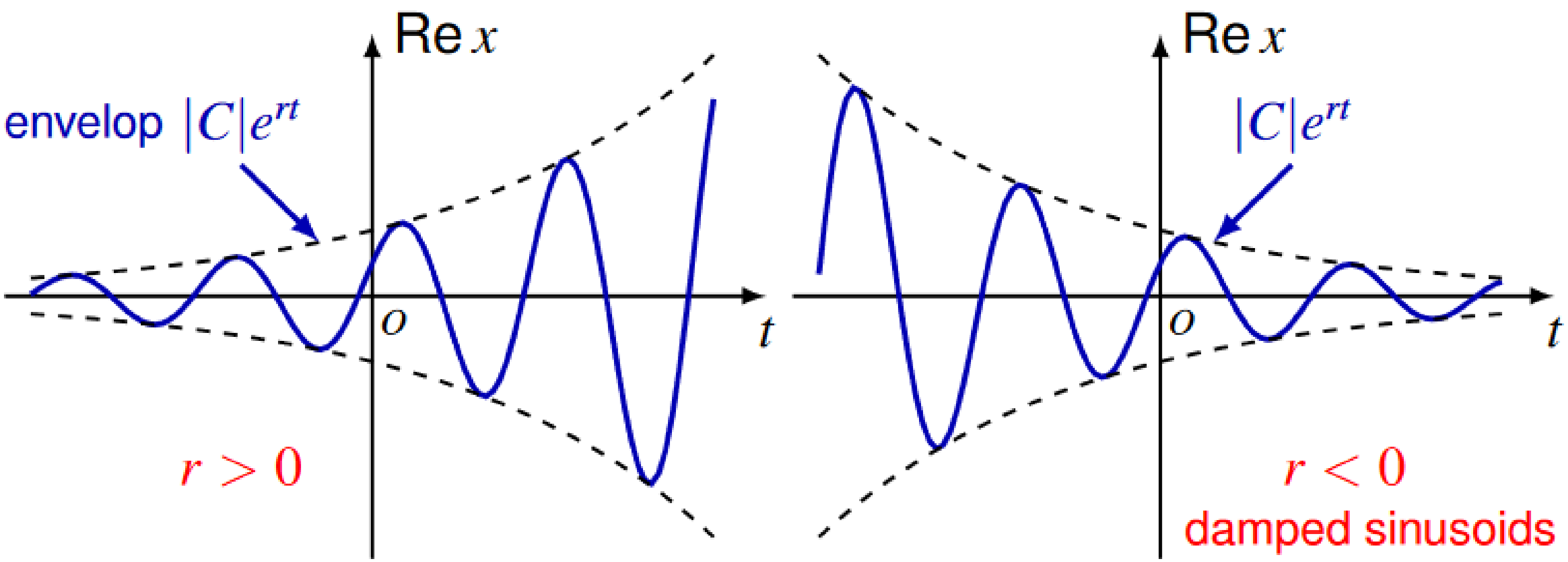
- Same periodicity
  - always periodic with fundamental frequency $|\omega_0|$
  - larger $|\omega_0|$, faster oscillation

General Complex Exponential Signals

$$x(t) = C \cdot e^{at}, \text{ where } C = |C|e^{j\phi},\ a = r + j\omega_0$$

$$\Downarrow$$

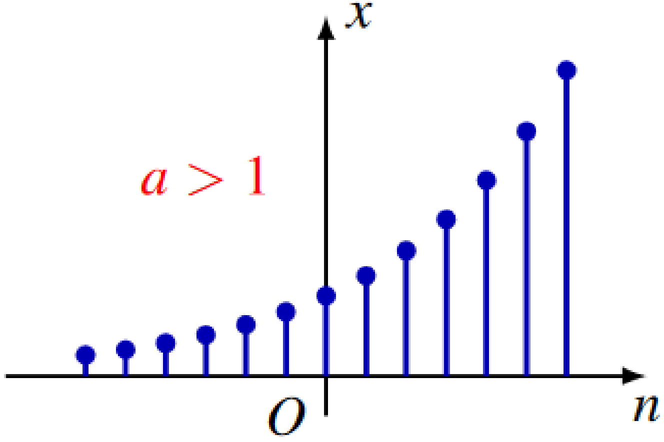
$$x(t) = |C|e^{rt}e^{j(\omega_0 t+\phi)} = |C|e^{rt}\cos(\omega_0 t + \phi) + j|C|e^{rt}\sin(\omega_0 t + \phi)$$

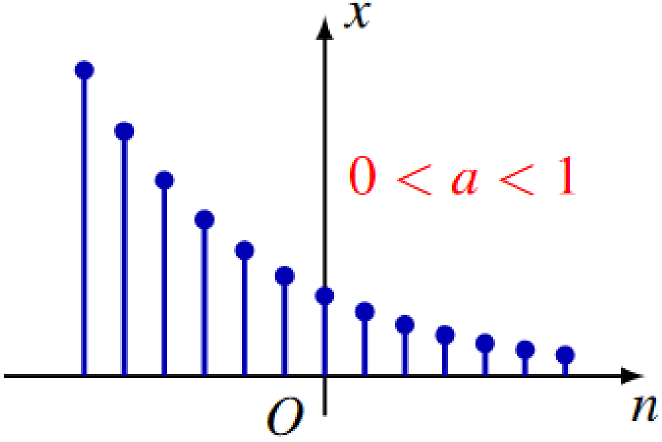


DT Complex Exponential Signals

$$x[n] = C \cdot \alpha^n = Ce^{\beta n}, \text{ where } C \in \mathbb{C}, \alpha = e^{\beta} \in \mathbb{C}$$

- **Real exponential signals:** $C \in \mathbb{R}$, $\boldsymbol{a} \in \mathbb{R}$ (but $\beta \in \mathbb{C}$ **!**)
  - 1. $\alpha > 1$: monotonically growing
  - 2. $0 < \alpha < 1$: monotonically decaying
  - 3. $\alpha = 0$: constant

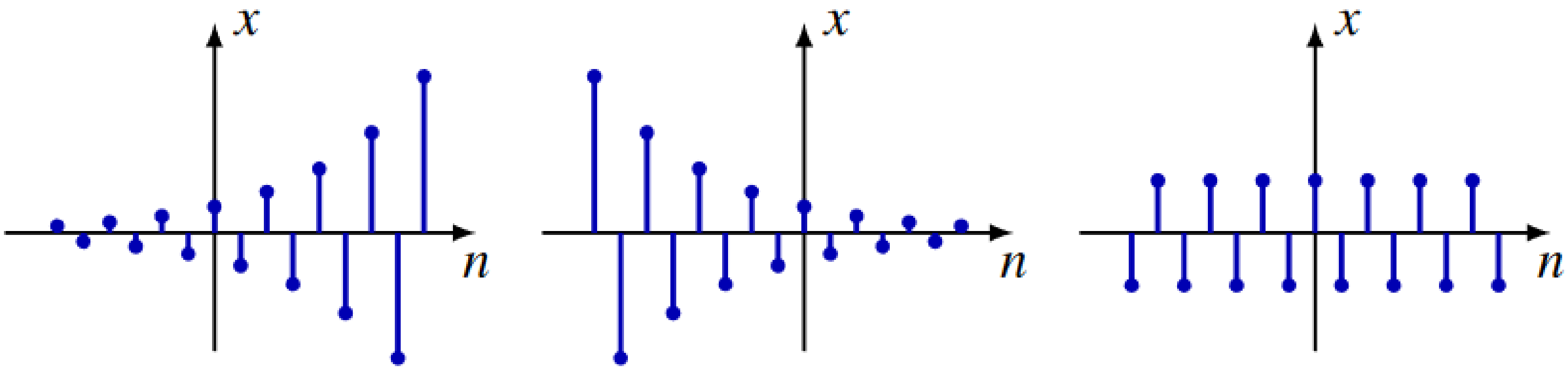




DT Complex Exponential Signals

$$x[n] = C \cdot \alpha^n = Ce^{\beta n}, \text{ where } C \in \mathbb{C}, \alpha = e^{\beta} \in \mathbb{C}$$

- **Real exponential signals:** $C \in \mathbb{R}$, $\alpha \in \mathbb{R}$ (but $\beta \in \mathbb{C}$ **!**)
  - 4. $\alpha < -1$: growing magnitude, alternating sign
  - 5. $-1 < \alpha < 0$: decaying magnitude, alternating sign
  - 6. $\alpha = -1$: constant magnitude, alternating sign ($\beta = j\pi$)



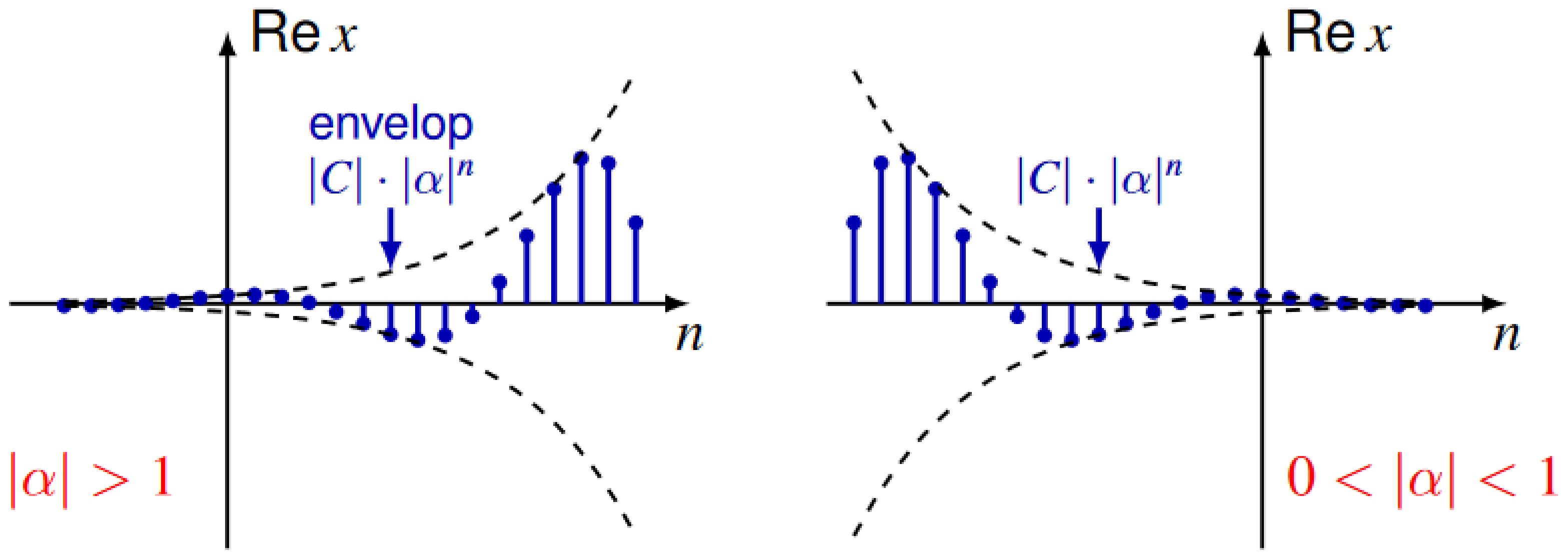
$\alpha < -1$ $\quad$ $-1 < \alpha < 0$ $\quad$ $\alpha = -1$

General Complex Exponential Signals

$$x[n] = C \cdot \alpha^n, \text{ where } C = |C|e^{j\phi}, \alpha = |\alpha|e^{j\omega_0}$$

$$\Downarrow$$

$$x[n] = |C|\alpha^n e^{j(\omega_0 n+\phi)} = |C||\alpha|^n \cos(\omega_0 n + \phi) + j|C||\alpha|^n \sin(\omega_0 n + \phi)$$



DT Complex Sinusoidal Signals

$$x[n] = |C|e^{j(\omega_0 n+\phi)} = |C|\cos(\omega_0 n + \phi) + j|C|\sin(\omega_0 n + \phi)$$

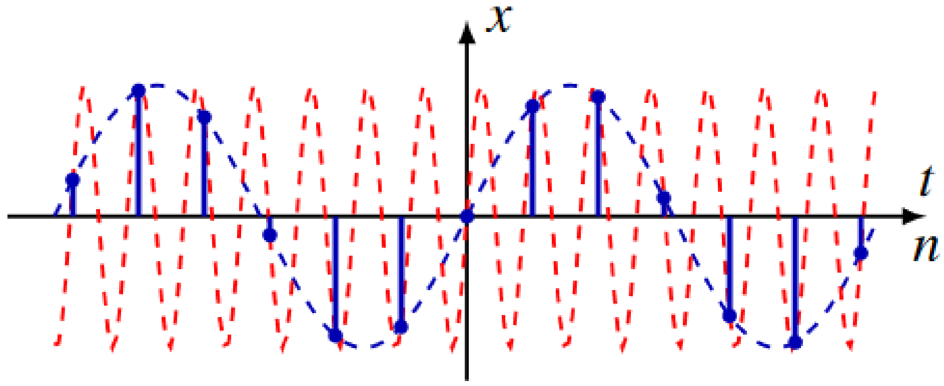
- Periodicity
  - periodic $\Leftrightarrow \omega_0 = \frac{2\pi \mathrm{k}}{N}$ for $k \in \mathbb{Z}, N \in \mathbb{Z}_+$
  - fundamental period $N_0 = N/\mathrm{gcd}(N, k)$
- Fundamental frequency
  - zero if $N_0 = 1$
  - $2\pi/N_0$ if $N_0 > 1$
- Example: $x[n] = e^{j3\pi n}$ has $N_0 = 2$, fundamental frequency π, not 3π! Note that $e^{j3\pi n} = e^{j\pi n}$.

DT Complex Exponential Signals

- **Aliasing**
  - $e^{j\omega_1 t} = e^{j\omega_2 t}, \forall t \in \mathbb{R} \Leftrightarrow \omega_1 = \omega_2$
  - $e^{j\omega_1 n} = e^{j\omega_2 n}, \forall n \in \mathbb{N} \Leftrightarrow \omega_1 = \omega_2 + 2k\pi, k \in \mathbb{Z}$

Frequencies differing by 2kπ yields the same discrete sinusoid

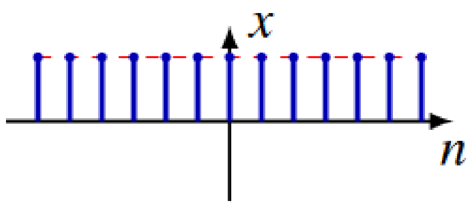
- **Example:** $\omega_1 = 1, \omega_2 = 1 + 2\pi$



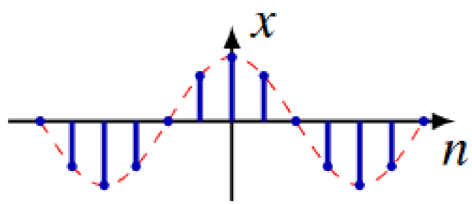
For DT signals, it suffices to consider frequencies on an interval of length 2π, e.g., $[0, 2\pi)$ or $(-\pi, \pi]$

DT Complex Exponential Signals

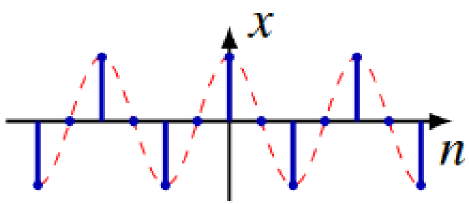
High frequencies around $(2k+1)\pi$, low frequencies around $2k\pi$



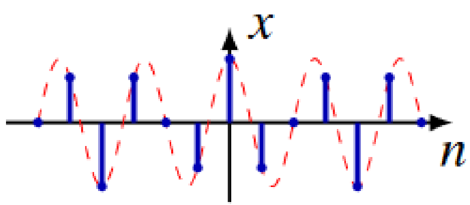
$x[n] = \cos(0 \cdot n) = 1$



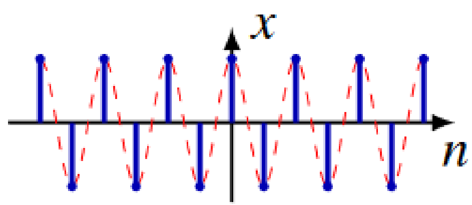
$x[n] = \cos(\pi n/4)$



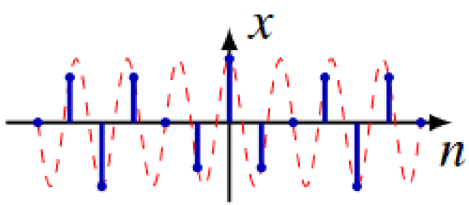
$x[n] = \cos(\pi n/2)$



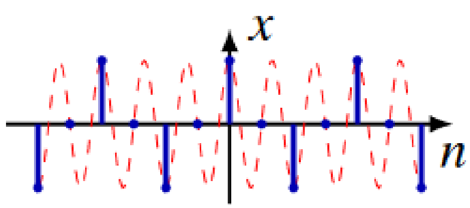
$x[n] = \cos(3\pi n/4)$



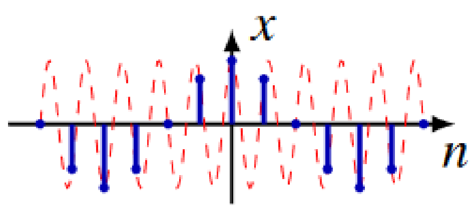
$x[n] = \cos(\pi n)$



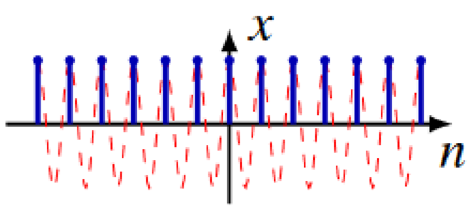
$x[n] = \cos(5\pi n/4)$



$x[n] = \cos(3\pi n/2)$



$x[n] = \cos(7\pi n/4)$



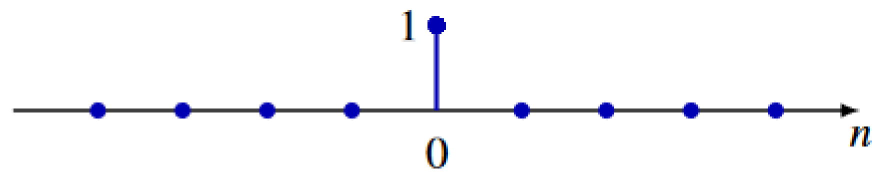
$x[n] = \cos(2\pi n) = 1$

Comparison on Periodic Properties of CT and DT Complex Exponentials and Sinusoids

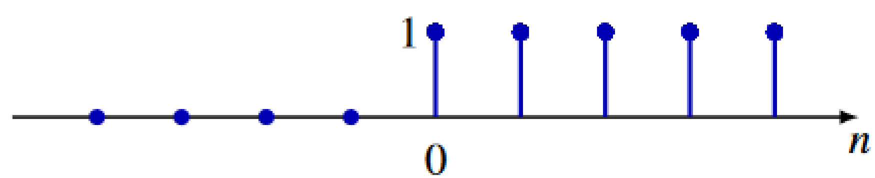
$x(t) = e^{j\omega_0 t}$	$x[n] = e^{j\omega_0 n}$
Distinct signals for distinct value of ω_0	Identical signals for values of ω_0 separated by multiples of 2π
Periodic for any choice of ω_0	Periodic only if $\omega_0 = 2\pi m/N$ for some integers $N > 0$ and m
Fundamental angular frequency ω_0	Fundamental angular frequency ω_0/n, if m and N do not have any factors in common
Fundamental period $2\pi/\omega_0$	Fundamental period $2\pi m/\omega_0$

DT Unit Impulse and Unit Step Sequences

- **Unit Impulse Sequence**
  - $\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$

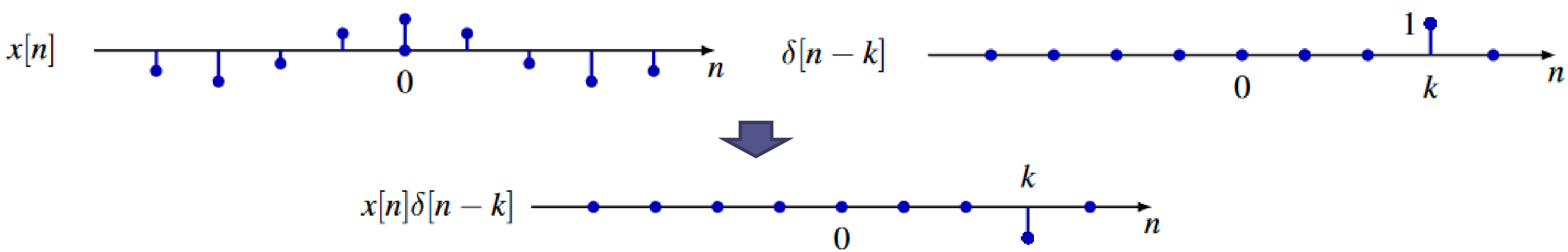


- **Unit Step Sequence**
  - $u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$



- **Relationship**
  - $\delta[n] = u[n] - u[n-1]$ first (backward) difference
  - $u[n] = \sum_{m=-\infty}^{n} \delta[m] = \sum_{k=0}^{\infty} \delta[n-k]$ running sum

- **Sampling Property**
  - $x[n] \cdot \delta[n] = x[0] \cdot \delta[n]$
  - $x[n] \cdot \delta[n-k] = x[k] \cdot \delta[n-k]$



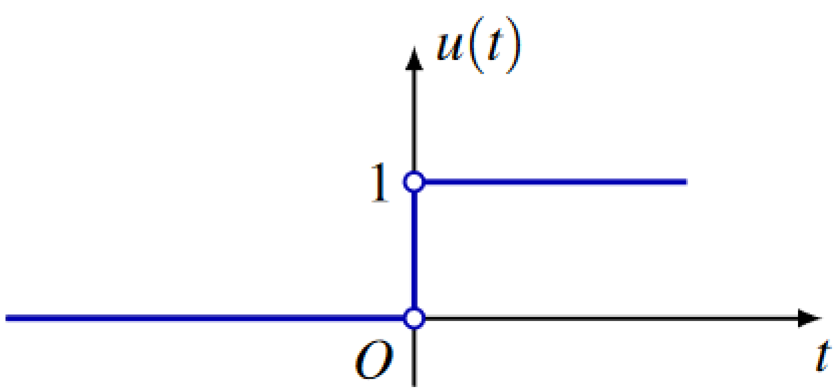
- **Signal representation by means of a series of delayed unit samples**
  - $x[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot \delta[n-k]$

CT Unit Step Function

- **Also called Heaviside (step) function**

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

  - undefined at $t = 0$
  - sometimes $u(0) = 0, 1, or\ 1/2$



Oliver Heaviside
(from Wikipedia)

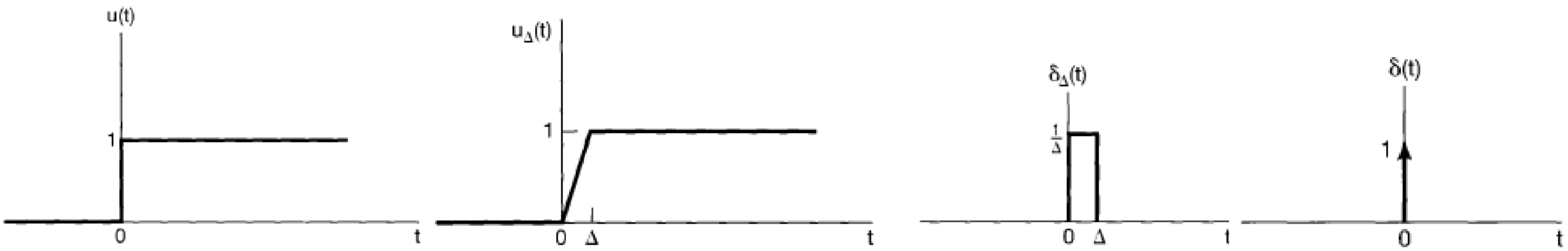
- **Recall that for DT unit step/impulse signals**
  - $\delta[n] = u[n] - u[n-1]$ —1st difference
  - $u[n] = \sum_{m=-\infty}^{n} \delta[m] = \sum_{k=0}^{\infty} \delta[n-k]$ —running sum

- **Does it exist in CT domain a $\delta(t)$ satisfying the following relationship?**
  - $\delta(t) = \frac{du(t)}{dt}$ —1st derivative
  - $u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$ —running sum ?

CT Unit Impulse Function

- **Define** $u_\Delta(t)$
  - rises from 0 to 1 in a very short interval Δ

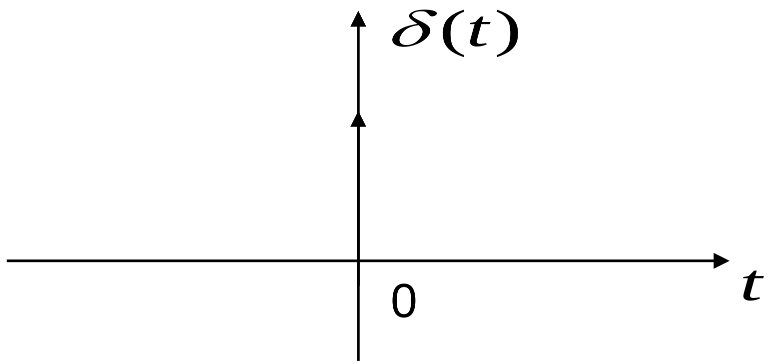
- **Then** $\delta_\Delta(t) = \dfrac{d(u_\Delta(t))}{dt}$, $\delta(t) = \lim\limits_{\Delta\to 0} \delta_\Delta(t)$



Notes: the amplitude of the signal $\delta(t)$ at $t = 0$ is infinite, but with a unit integral from $-\infty$ to $+\infty$, i.e., from 0^- to 0^+.

- **Also called Dirac delta function**

$$\begin{cases} \int_{-\infty}^{\infty} \delta(t)dt = 1 \\ \delta(t) = 0, \quad t \neq 0 \end{cases}$$




Paul Dirac
(from Wikipedia)

- **Physical models**
  - density of point mass/charge
  - impulse force

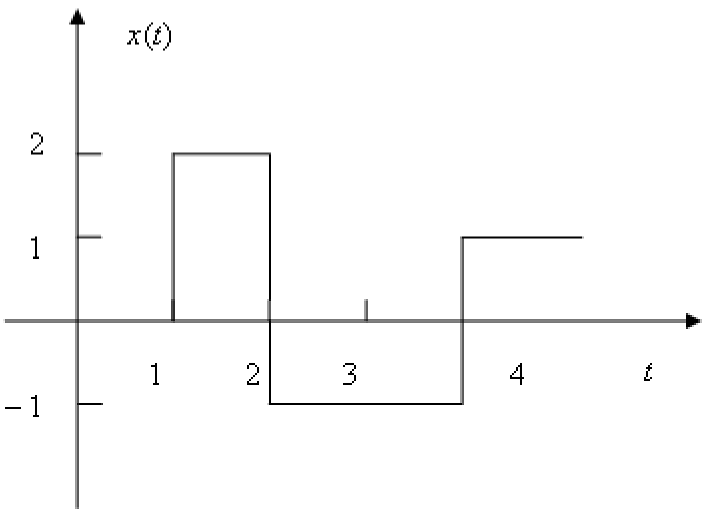
- **Relationship**
  - $\delta(t) = \frac{du(t)}{dt}$
  - $u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$

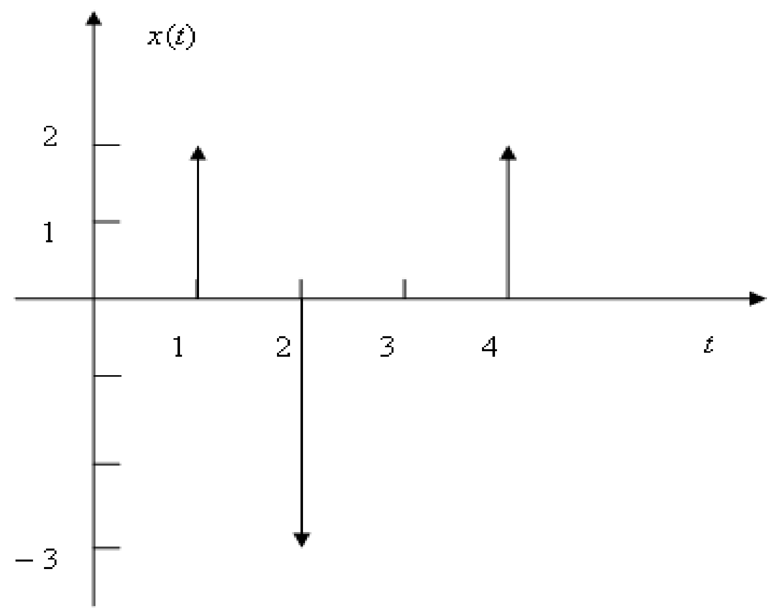
- **Sampling Property**
  - $x(t) \cdot \delta(t) = x(0) \cdot \delta(t)$
  - $x(t) \cdot \delta(t - t_0) = x(t_0) \cdot \delta(t - t_0)$

- **Scaling Property**
  - $\frac{d(ku(t))}{dt} = k\delta(t)$

Question: can we represent $x(t)$ by using a series of unit samples as that for DT signal?

- **Example:**
  - Derive the 1st derivative of the following $x(t)$
    - a) $x(t) = 2u(t-1) - 3u(t-2) + 2u(t-4)$
    - b) $\frac{dx(t)}{dt} = 2\delta(t-1) - 3\delta(t-2) + 2\delta(t-4)$

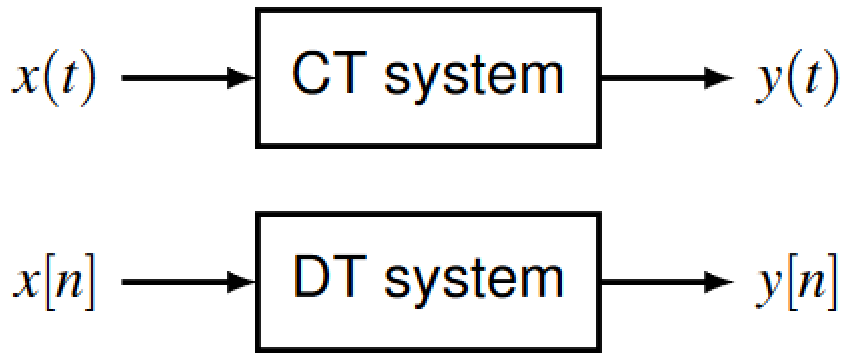




- **Example:**
  - Calculate the following signals/values
    - a) $(t^2-1)\delta(t-2)$
    - b) $\int_{-3}^{3}(t^2-1)\delta(t-2)\,dt$
    - c) $x[n-3]\delta[n+1]$
    - d) $\int_{-3}^{t}(\tau^2-1)\delta(\tau-2)\,d\tau$

Systems

- A system takes some input and produces some output



- **Example:** balance of your bank account
  - Input $x[n]$: net deposit on the n-th day
  - Output $y[n]$: balance at the end of the n-th day
  - Input-output relation:

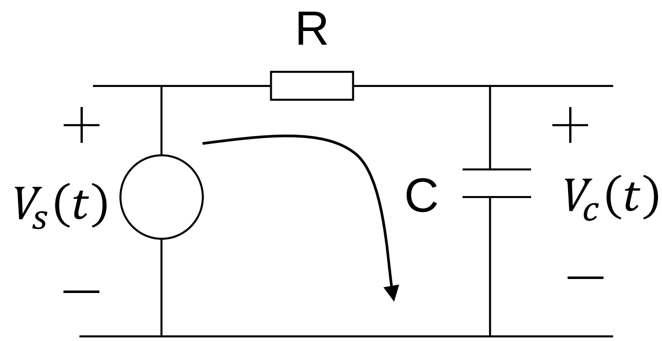
$$y[n] \;=\; (1+r)y[n-1] \;+ x[n], \qquad r \text{ interest rate}$$

System Modeling

- **RLC Circuit**

$$\because i(t) = \frac{V_s(t) - V_c(t)}{R} \quad i(t) = C \cdot \frac{dV_c(t)}{dt}$$

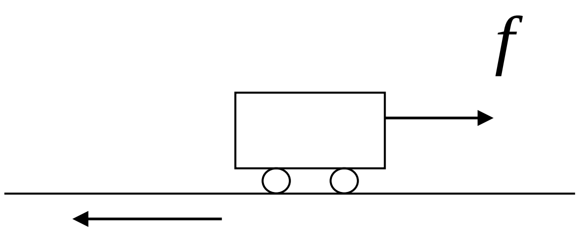
$$\therefore \frac{V_C}{dt} + \frac{1}{RC} V_C(t) = \frac{1}{RC} V_s(t)$$



- **Mechanism System**

$$\because \frac{dv(t)}{dt} = \frac{1}{m}[f(t) - \rho v(t)]$$

$$\therefore \frac{dv(t)}{dt} + \frac{\rho}{m} v(t) = \frac{f(t)}{m}$$



System Modeling

- **Observations**

$$\frac{dy(t)}{dt} + ay(t) = bx(t)$$

$$ay[n-1] + y[n] = bx[n]$$

  - Constant coefficient differential/difference equations
  - Very different physical systems may
    - be modeled mathematically in very similar ways
    - have very similar mathematical descriptions

Typical Systems

- **Amplifier**

$$y(t) = cx(t)$$

- **Adder**

$$y(t) = x_1(t) + x_2(t)$$

- **Multiplier**

$$y(t) = x_1(t) \cdot x_2(t)$$

- **Differentiator/Difference**

$$y(t) = dx(t)/dt, \qquad y[n] = x[n] - x[n-1]$$

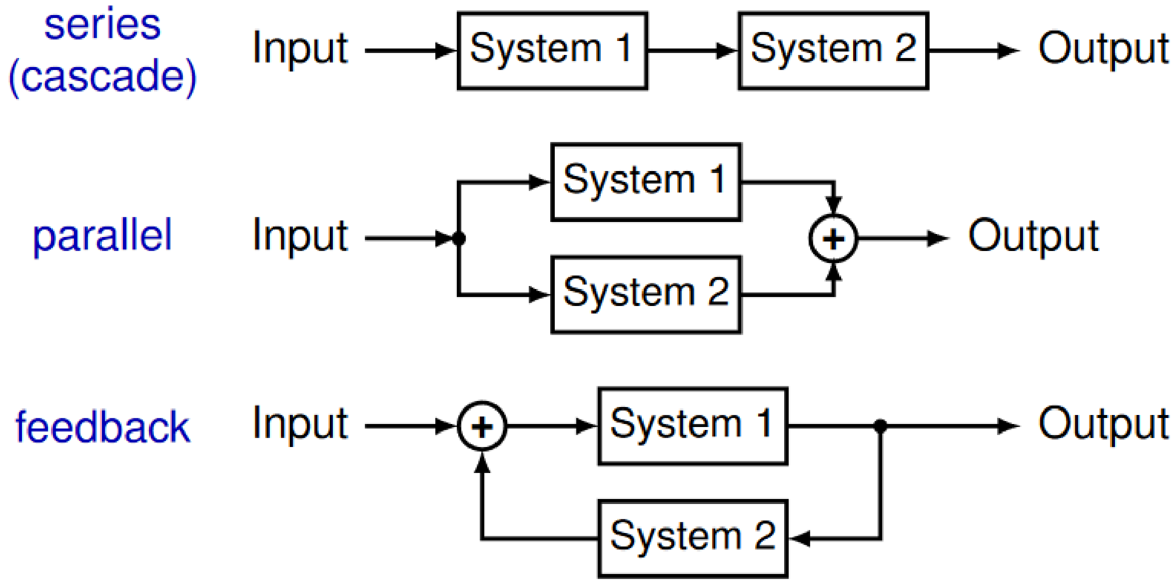
- **Integrator/Accumulator**

……

System Interconnections

- **Concept**
  - Build a complex system from interconnected subsystems
  - Scope of subsystem depends on level of abstraction

- **Basic Types of Interconnections**



Memory

- **Systems with memory**
  - if the current output of the system is dependent on future and/or past values of the inputs and/or outputs, e.g.,
  - Capacitor system:

$$u(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)\, d\tau,\ \ y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)\, d\tau$$

  - Accumulator system:

$$y[n] = \sum_{k=-\infty}^{n} x[k],\ \ y[n] = \sum_{k=-\infty}^{n-1} x[k] + x[n] = y[n-1] + x[n]$$

- **Memoryless systems:**
  - if the current output of the system is dependent on the input at the same time, e.g.,
  - Identity system:

$$y(t) = x(t), y[n] = x[n]$$

- **Example:**
  - Determine the memory property of the following systems:
    - a) amplifier, adder, multiplier;
    - b) integrator, accumulator;
    - c) differentiator;
    - d) time reversal, time scalar;
    - e) decimator, interpolator.

Invertibility

- **Inverse systems**
  - distinct inputs lead to distinct outputs, e.g.,

$$y(t) = 2x(t) \rightarrow w(t) = \frac{1}{2}y(t)$$

- **Non-inverse systems**
  - distinct inputs may lead to the same outputs, e.g.,

$$y(t) = x^2(t), \qquad y[n] = 0$$

- **Importance of the concept**
  - encoding for channel coding or lossless compress

Causality

- **A system is causal**
  - if output at any time t depends only on input values up to t
  - i.e., output does not anticipate future values of the input
- **Notes:**
  - All real-time physical systems are causal
    - because time only moves forward, effect occurs after cause
    - e.g., imagine if you own a non-causal system whose output depends on tomorrow's stock price.
  - Causality does not apply to spatially varying signals
    - one can move both left and right, up and down
  - Causality does not apply to recorded signals
    - e.g., taped sports games vs. live show.

Causality

- **For a causal system** $x(t) \rightarrow y(t)$

$$x_1(t) \rightarrow y_1(t), \quad x_2(t) \rightarrow y_2(t)$$

$$\text{if} \qquad x_1(t) = x_2(t), \ \forall t \leq t_0$$

$$\text{then} \qquad y_1(t) = y_2(t), \ \forall t \leq t_0$$

  - If two inputs to a causal system are identical up to some point in time t_0, the corresponding outputs are also equal up to the same time t_0

- **Example:**
  - Determine the causality of the following signals
    - $y(t) = x^2(t-1)$
      - e.g., $y(5)$ depends on $x(4)$ … causal
    - $y(t) = x(t+1)$
      - e.g., $y(5) = x(6)$ depends on future … noncausal
    - $y[n] = x[-n]$
      - e.g., $y(5) = x(-5)$, but $y[-5] = x[5]$ depends on future … noncausal
    - $y[n] = \left(\frac{1}{2}\right)^{n+1} x^3[n-1]$
      - e.g., $y(5)$ depends on $x(4)$ … causal

Linearity

- **A system** $x(t) \rightarrow y(t)$ **is linear**
  - if for any two input-output: $x_1(t) \rightarrow y_1(t)$, $x_2(t) \rightarrow y_2(t)$
  - the additivity and scaling properties hold

    additivity: $x_1(t) + x_2(t) \rightarrow y_1(t) + y_2(t)$

    scaling: $ax_1(t) \rightarrow ay_1(t)$
  - or equivalently, the superposition property holds

    superposition: $ax_1(t) + bx_2(t) \rightarrow ay_1(t) + by_2(t)$

- **Example:**
  - $y[n] = x_2[n]$ nonlinear, causal
  - $y(t) = x(2t)$ linear, non-causal

Linearity

- **Many systems are nonlinear**
  - Examples: many circuit elements (e.g., diodes), dynamics of aircraft, econometric models, …
- **But why we study linear systems?**
  - Linear models represent accurate representations of behavior of many systems
    - e.g., linear resistors, capacitors, other examples given previously
  - We can often linearize models to examine “small signal” perturbations around “operating points”
  - Linear systems are analytically tractable, providing basis for important tools and considerable insight

Time-invariance

- **A system is time-invariant**
  - if its behavior does not depend on what time it is;
  - i.e., time shift in input results in identical time shift in output

- **Mathematical definition**
  - For a DT system: A system $x[n] \rightarrow y[n]$ is time-invariant if for any input x[n] and any time shift n_0,

    if $\quad x[n] \rightarrow y[n]$

    then $\quad x[n-n_0] \rightarrow y[n-n_0]$
  - Similarly for a CT time-invariant system,

    if $\quad x(t) \rightarrow y(t)$

    then $\quad x\left(t-t_0\right) \rightarrow y(t-t_0)$

- **Example:**
  - Consider the time-invariance property of the following systems:
    - $y[n] = nx[n]$ time-varying
    - $y(t) = x_2(t+1)$ time-invariant

- **Example:**
  - For a time-invariant system $x(t) \to y(t)$,
    - if input is periodic with T, $x(t) = x(t+T)$, then the output is also periodic with T, i.e.,

$$y(t) = y(t + T)$$

- **Example:**
  - $y(t) = \cos(x(t))$ time-invariant

- **Example**
  - amplitude modulator:
    - $y(t) = x(t)\cos\omega t$ time-varying

Stability

- **A System is bounded-input bounded-output (BIBO) stable**
  - if outputs are bounded for all bounded inputs

- **Example:**
  - When $|x(t)| \leq B$, determine whether or not the following systems are stable?
  - a) $y(t) = t \cdot x(t)$, unstable
  - b) $y(t) = e^{x(t)}$, stable

Linear Time-Invariant (LTI) Systems

- **Using superposition and time-invariant properties**
  - if response of an LTI system to some inputs ("basic signals") is known, we then actually know the response to many inputs
  - if $x_k[n] \rightarrow y_k[n]$

    then $\sum_k a_k x_k[n] \rightarrow \sum_k a_k y_k[n]$

- **Characteristics of "basic signals"**
  - can represent rich classes of signals as linear combinations of these building block signals
  - response of LTI Systems to these basic signals are both simple and insightful

Q & A

Many Thanks